Linear independence I — concept and geometric meaning

Practice problems

1. Determine if the vectors $v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ are linearly independent.
2. Check the linear independence of $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, and $v_3 = (1, 1, 0)$.
3. Determine if the vectors $v_1 = (1, 2, 1)$, $v_2 = (0, 1, 1)$, and $v_3 = (1, 0, -1)$ are linearly independent using the determinant.

Solutions

1. Determine if the vectors $v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ are linearly independent.

   - Check if v_2 is a scalar multiple of v_1.
   - Observe that $2 \times 2 = 4$ and $2 \times 3 = 6$.
   - Thus, $v_2 = 2v_1$.
   - Since one vector is a multiple of the other, they are linearly dependent.

   Answer: Linearly dependent

2. Check the linear independence of $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, and $v_3 = (1, 1, 0)$.

   - Stack the vectors as rows of a matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.
   - Perform row reduction: $R_3 \leftarrow R_3 - R_1$ gives $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.
   - Then $R_3 \leftarrow R_3 - R_2$ gives $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
   - The presence of a zero row indicates linear dependence.

   Answer: Linearly dependent

3. Determine if the vectors $v_1 = (1, 2, 1)$, $v_2 = (0, 1, 1)$, and $v_3 = (1, 0, -1)$ are linearly independent using the determinant.

   - Form the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.
   - Compute the determinant: $1(1(-1) - 1(0)) - 2(0(-1) - 1(1)) + 1(0(0) - 1(1))$.
   - This equals $1(-1) - 2(-1) + 1(-1) = -1 + 2 - 1 = 0$.
   - Since the determinant is zero, the vectors are linearly dependent.

   Answer: Linearly dependent