

The column space of a matrix

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Practice problems

1. Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is b in the column space of A ? Justify your answer.
2. Find a basis for the column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$.
3. If a 4×3 matrix A has rank 2, what is the geometric nature of its column space in $\mathbb{R}^4$?

Solutions

1. Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is b in the column space of A ? Justify your answer.

- We check if $x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.
- From the first row: $x_1 + 2x_2 = 1$.
- From the second row: $3x_1 + 4x_2 = 2$.
- Subtracting 3 times the first from the second: $(3-3)x_1 + (4-6)x_2 = 2-3$, so $-2x_2 = -1$, which means $x_2 = 0.5$.
- Substituting back: $x_1 + 2(0.5) = 1$, so $x_1 = 0$.
- Check the third row: $5(0) + 6(0.5) = 3$. This matches the third component of b .

Answer: Yes, b is in the column space because $b = 0v_1 + 0.5v_2$.

2. Find a basis for the column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$.

- Perform row reduction: $R_3 \rightarrow R_3 - R_1$ gives $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.
- Next, $R_3 \rightarrow R_3 - R_2$ gives $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
- The pivots are in the first and second columns.
- The basis for the column space consists of the first and second columns of the original matrix.

Answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

3. If a 4×3 matrix A has rank 2, what is the geometric nature of its column space in $\mathbb{R}^4$?
- The column space is a subspace of $\mathbb{R}^4$ because the columns have 4 components.
 - The dimension of the column space is equal to the rank of the matrix.
 - Since the rank is 2, the dimension of the column space is 2.

Answer: The column space is a 2-dimensional plane passing through the origin in 4-dimensional space.