The determinant as area/volume scaling factor

Lesson 44 · Module 3 — Determinants · Linear Algebra · dailymaths.uk/linalg

Practice problems

1. A region in the plane has an area of 7. It is transformed by the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$. What is the area of the transformed region?
2. Does the transformation represented by $M = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ preserve the orientation of the plane? Does it preserve area?
3. A transformation T scales all areas by a factor of 4 and reverses the orientation of the plane. If the matrix is $A = \begin{bmatrix} a & 1 \\ 2 & 1 \end{bmatrix}$, find the value of a.

Solutions

1. A region in the plane has an area of 7. It is transformed by the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$. What is the area of the transformed region?

   - First, compute the determinant of A: $\det(A) = (4 \times 1) - (1 \times 2) = 4 - 2 = 2$.
   - The area scaling factor is the absolute value of the determinant, which is $|2| = 2$.
   - Multiply the original area by the scaling factor: $7 \times 2 = 14$.

   Answer: 14

2. Does the transformation represented by $M = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ preserve the orientation of the plane? Does it preserve area?

   - Compute the determinant: $\det(M) = (2 \times 3) - (5 \times 1) = 6 - 5 = 1$.
   - Since the determinant is positive, the orientation is preserved.
   - Since the absolute value of the determinant is 1, the area is preserved.

   Answer: Yes to both

3. A transformation T scales all areas by a factor of 4 and reverses the orientation of the plane. If the matrix is $A = \begin{bmatrix} a & 1 \\ 2 & 1 \end{bmatrix}$, find the value of a.

   - The area scaling factor is $|\det(A)| = 4$.
   - Since the orientation is reversed, $\det(A)$ must be -4.
   - Set up the equation: $(a \times 1) - (1 \times 2) = -4$.
   - Solve for a: $a - 2 = -4$, so $a = -2$.

   Answer: -2