Block matrices and partitioned multiplication

Lesson 42 · Module 2 — Matrix Algebra · Linear Algebra · dailymaths.uk/linalg

Practice problems

1. Given $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, partition both into 2×2 blocks and compute the top-left block C_{11} of the product $C = AB$.
2. If A and B are both 4×4 block diagonal matrices with 2×2 blocks A_{11}, A_{22} and B_{11}, B_{22}, what is the resulting matrix $C = AB$ in terms of these blocks?
3. Let A be a 2×4 matrix partitioned as $[A_{11}\ A_{12}]$ and B be a 4×2 matrix partitioned as $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$, where all blocks are 2×2. If $A_{11} = I$, $A_{12} = I$, $B_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $B_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, compute the resulting 2×2 matrix C.

Solutions

1. Given $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, partition both into 2×2 blocks and compute the top-left block C_{11} of the product $C = AB$.

   - Identify $A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and $B_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
   - Apply the block formula: $C_{11} = A_{11}B_{11} + A_{12}B_{21}$.
   - Compute $A_{11}B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
   - Compute $A_{12}B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
   - Sum the results: $C_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

   Answer: \beginbmatrix 2 & 0 \\ 0 & 2 \endbmatrix

2. If A and B are both 4×4 block diagonal matrices with 2×2 blocks A_{11}, A_{22} and B_{11}, B_{22}, what is the resulting matrix $C = AB$ in terms of these blocks?

   - A block diagonal matrix has the form $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$.
   - The product of two such matrices is $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$.
   - Using block multiplication: $C_{11} = A_{11}B_{11} + 0 \cdot 0 = A_{11}B_{11}$.
   - Similarly, $C_{12} = A_{11} \cdot 0 + 0 \cdot B_{22} = 0$.
   - And $C_{21} = 0 \cdot B_{11} + A_{22} \cdot 0 = 0$.
   - Finally, $C_{22} = 0 \cdot 0 + A_{22}B_{22} = A_{22}B_{22}$.

   Answer: \beginbmatrix A_11B_11 & 0 \\ 0 & A_22B_22 \endbmatrix

3. Let A be a 2×4 matrix partitioned as $[A_{11}\ A_{12}]$ and B be a 4×2 matrix partitioned as $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$, where all blocks are 2×2. If $A_{11} = I$, $A_{12} = I$, $B_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $B_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, compute the resulting 2×2 matrix C.

   - The formula for this product is $C = A_{11}B_{11} + A_{12}B_{21}$.
   - Substitute the given blocks: $C = I \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + I \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
   - Since I is the identity, $IM = M$ for any matrix M.
   - So, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
   - Perform the addition: $C = \begin{bmatrix} 1+0 & 2+1 \\ 3+1 & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 4 \end{bmatrix}$.

   Answer: \beginbmatrix 1 & 3 \\ 4 & 4 \endbmatrix