LU decomposition II — solving systems efficiently

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Practice problems

1. Using the matrices $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, solve the system $Ax = b$ for $b = \begin{bmatrix} 5 \\ 10 \\ 23 \end{bmatrix}$.
2. Given $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$, find the determinant of $A = LU$.
3. If a 3×3 matrix A has an LU decomposition where U has diagonal elements $2, 0, 5$, what can you conclude about the system $Ax = b$?

Solutions

1. Using the matrices $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, solve the system $Ax = b$ for $b = \begin{bmatrix} 5 \\ 10 \\ 23 \end{bmatrix}$.

   - Step 1: Solve $Ly = b$. $y_1 = 5$. $2(5)+y_2 = 10 \implies y_2 = 0$. $3(5)+4(0)+y_3 = 23 \implies y_3 = 8$. So $y = [5, 0, 8]^T$.
   - Step 2: Solve $Ux = y$. $1x_3 = 8 \implies x_3 = 8$. $3x_2 + 2(8) = 0 \implies 3x_2 = -16 \implies x_2 = -16/3$. $2x_1 + (-16/3) + 8 = 5 \implies 2x_1 = 5 - 8 + 16/3 = -3 + 16/3 = 7/3 \implies x_1 = 7/6$.

   Answer: x = [7/6, -16/3, 8]^T

2. Given $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$, find the determinant of $A = LU$.

   - The determinant of A is the product of the diagonal elements of U.
   - $\det(A) = 2 \times 4 = 8$.

   Answer: 8

3. If a 3×3 matrix A has an LU decomposition where U has diagonal elements $2, 0, 5$, what can you conclude about the system $Ax = b$?

   - The determinant of A is the product of the diagonal elements of U.
   - $\det(A) = 2 \times 0 \times 5 = 0$.
   - Since the determinant is zero, the matrix A is singular.

   Answer: The matrix A is singular, meaning the system may have no solution or infinitely many solutions.
