Homogeneous systems and why they matter

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Practice problems

1. Determine if the system $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$ has only the trivial solution or infinitely many solutions.
2. Given that $\mathbf{v}_1 = [1, 2, -1]^T$ and $\mathbf{v}_2 = [0, 1, 3]^T$ are solutions to a homogeneous system $A\mathbf{x} = \mathbf{0}$, find a third non-zero solution $\mathbf{v}_3$.
3. Find the general solution in parametric vector form for the system $x+2y-z = 0$ and $2x+4y-2z = 0$.

Solutions

1. Determine if the system $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$ has only the trivial solution or infinitely many solutions.

   - Write the augmented matrix and perform row reduction.
   - Row 2 minus 2 times Row 1 gives [0 0 0].
   - Row 3 minus Row 1 gives $[0 \ -1 \ -1]$.
   - The RREF is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
   - There is a free variable z, so there are infinitely many solutions.

   Answer: Infinitely many solutions

2. Given that $\mathbf{v}_1 = [1, 2, -1]^T$ and $\mathbf{v}_2 = [0, 1, 3]^T$ are solutions to a homogeneous system $A\mathbf{x} = \mathbf{0}$, find a third non-zero solution $\mathbf{v}_3$.

   - Use the superposition principle: any linear combination of solutions is a solution.
   - Let $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.
   - Compute the sum: $[1 + 0, 2 + 1, -1 + 3]^T = [1, 3, 2]^T$.

   Answer: $[1, 3, 2]^T$ (or any other linear combination)

3. Find the general solution in parametric vector form for the system $x + 2y - z = 0$ and $2x + 4y - 2z = 0$.

   - The second equation is a multiple of the first, so we have only one independent equation: $x + 2y - z = 0$.
   - Identify x as the pivot variable and y, z as free variables.
   - Solve for x: $x = -2y + z$.
   - Write in vector form: $\mathbf{x} = y[-2, 1, 0]^T + z[1, 0, 1]^T$.

   Answer: $\mathbf{x} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
