

Logistic Growth

Lesson 96 · Module 4 — Applications of Integration · The Calculus Course · dailymaths.uk/course

Practice problems

1. A population of fish in a pond follows the logistic model with a carrying capacity of $L = 2000$ and a growth constant $k = 0.1$ per month. If the initial population is $y(0) = 500$, find the population after $t = 10$ months.
2. A city's population grows according to the logistic equation $\frac{dy}{dt} = 0.02y(1 - \frac{y}{100000})$. If the current population is 20,000, what is the intrinsic growth rate k and the carrying capacity L ?
3. Show that for the logistic equation $\frac{dy}{dt} = ky(1 - \frac{y}{L})$, the maximum growth rate occurs when the population is exactly half of the carrying capacity, $y = L/2$.

Solutions

1. A population of fish in a pond follows the logistic model with a carrying capacity of $L = 2000$ and a growth constant $k = 0.1$ per month. If the initial population is $y(0) = 500$, find the population after $t = 10$ months.

- Use the formula $y(t) = \frac{L}{1 + Ae^{-kt}}$.
- Find A using $y(0) = 500$: $500 = \frac{2000}{1+A} \implies 1 + A = 4 \implies A = 3$.
- Substitute $t = 10$: $y(10) = \frac{2000}{1+3e^{-0.1(10)}} = \frac{2000}{1+3e^{-1}}$.
- Calculate $e^{-1} \approx 0.3679$, so $y(10) \approx \frac{2000}{1+1.1037} \approx \frac{2000}{2.1037} \approx 950.7$.

Answer: $y(10) \approx 951$

2. A city's population grows according to the logistic equation $\frac{dy}{dt} = 0.02y(1 - \frac{y}{100000})$. If the current population is 20,000, what is the intrinsic growth rate k and the carrying capacity L ?

- Compare the given equation $\frac{dy}{dt} = 0.02y(1 - \frac{y}{100000})$ to the standard form $\frac{dy}{dt} = ky(1 - \frac{y}{L})$.
- The coefficient k is the value multiplying the y term outside the parentheses, so $k = 0.02$.
- The value L is the denominator in the fraction inside the parentheses, so $L = 100,000$.

Answer: $k = 0.02, L = 100,000$

3. Show that for the logistic equation $\frac{dy}{dt} = ky(1 - \frac{y}{L})$, the maximum growth rate occurs when the population is exactly half of the carrying capacity, $y = L/2$.

- The growth rate is given by the function $f(y) = ky - \frac{k}{L}y^2$.
- To find the maximum of $f(y)$, take the derivative with respect to y and set it to zero: $f'(y) = k - \frac{2k}{L}y$.
- Set $k - \frac{2k}{L}y = 0$.
- Divide by k (assuming $k \neq 0$): $1 - \frac{2y}{L} = 0 \implies \frac{2y}{L} = 1 \implies y = L/2$.

Answer: $y = L/2$