

Gabriel's horn: finite volume, infinite surface

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Practice problems

1. Consider the solid formed by rotating $f(x) = \frac{1}{x^2}$ for $x \geq 1$ around the x -axis. Does the volume V converge? If so, find its value.
2. For the solid in the previous problem ($f(x) = \frac{1}{x^2}$ for $x \geq 1$), use a comparison to determine if the surface area S is finite or infinite.
3. Suppose we rotate $f(x) = \frac{1}{\sqrt{x}}$ for $x \geq 1$ around the x -axis. Determine if the volume V is finite or infinite.

Solutions

1. Consider the solid formed by rotating $f(x) = \frac{1}{x^2}$ for $x \geq 1$ around the x -axis. Does the volume V converge? If so, find its value.

- Set up the volume integral: $V = \int_1^\infty \pi \left(\frac{1}{x^2}\right)^2 dx = \int_1^\infty \frac{\pi}{x^4} dx$.
- Evaluate as a limit: $\lim_{b \rightarrow \infty} \int_1^b \pi x^{-4} dx$.
- Find the antiderivative: $\pi \left[\frac{x^{-3}}{-3}\right]_1^b = \pi \left[-\frac{1}{3b} - \left(-\frac{1}{3}\right)\right]$.
- Take the limit as $b \rightarrow \infty$: $\pi \left(0 + \frac{1}{3}\right) = \frac{\pi}{3}$.

Answer: $V = \frac{\pi}{3}$

2. For the solid in the previous problem ($f(x) = \frac{1}{x^2}$ for $x \geq 1$), use a comparison to determine if the surface area S is finite or infinite.

- The surface area formula is $S = \int_1^\infty 2\pi \frac{1}{x^2} \sqrt{1 + \left(-\frac{2}{x^3}\right)^2} dx$.
- Note that $\sqrt{1 + \frac{4}{x^6}} > 1$ for all $x \geq 1$.
- Therefore, $S > \int_1^\infty \frac{2\pi}{x^2} dx$.
- The integral $\int_1^\infty \frac{1}{x^2} dx$ is a p -integral with $p = 2$, which converges.
- Since the integrand $\frac{1}{x^2} \sqrt{1 + \frac{4}{x^6}}$ behaves like $\frac{1}{x^2}$ for large x , the surface area actually converges.

Answer: The surface area is finite.

3. Suppose we rotate $f(x) = \frac{1}{\sqrt{x}}$ for $x \geq 1$ around the x -axis. Determine if the volume V is finite or infinite.

- Set up the volume integral: $V = \int_1^\infty \pi \left(\frac{1}{\sqrt{x}}\right)^2 dx$.
- Simplify the integrand: $V = \int_1^\infty \frac{\pi}{x} dx$.
- Evaluate the improper integral: $\lim_{b \rightarrow \infty} [\pi \ln x]_1^b$.
- The limit $\lim_{b \rightarrow \infty} \pi \ln b$ diverges to infinity.

Answer: $V = \infty$