Centre of mass and centroids

Practice problems

1. Find the centroid of the region bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$.
2. Find the centroid of the region bounded by $y = x^2$ and $y = x$.
3. Find the centroid of the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

Solutions

1. Find the centroid of the region bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$.

   - Calculate area: $A = \int_0^4 x^{1/2}\,dx = [\frac{2}{3}x^{3/2}]_0^4 = \frac{2}{3}(8) = \frac{16}{3}$.
   - Calculate M_y: $M_y = \int_0^4 x(x^{1/2})\,dx = \int_0^4 x^{3/2}\,dx = [\frac{2}{5}x^{5/2}]_0^4 = \frac{2}{5}(32) = \frac{64}{5}$.
   - Calculate M_x: $M_x = \int_0^4 \frac{1}{2}(\sqrt{x})^2\,dx = \int_0^4 \frac{1}{2}x\,dx = [\frac{1}{4}x^2]_0^4 = 4$.
   - Find $\bar{x} = \frac{64/5}{16/3} = \frac{64}{5} \cdot \frac{3}{16} = \frac{12}{5} = 2.4$.
   - Find $\bar{y} = \frac{4}{16/3} = 4 \cdot \frac{3}{16} = \frac{3}{4} = 0.75$.

   Answer: (\frac125, \frac34)

2. Find the centroid of the region bounded by $y = x^2$ and $y = x$.

   - Intersection points are $x = 0$ and $x = 1$.
   - Area $A = \int_0^1 (x - x^2)\,dx = [\frac{1}{2}x^2 - \frac{1}{3}x^3]_0^1 = \frac{1}{6}$.
   - Moment $M_y = \int_0^1 x(x - x^2)\,dx = [\frac{1}{3}x^3 - \frac{1}{4}x^4]_0^1 = \frac{1}{12}$.
   - Moment $M_x = \int_0^1 \frac{1}{2}(x^2 - (x^2)^2)\,dx = \frac{1}{2}[\frac{1}{3}x^3 - \frac{1}{5}x^5]_0^1 = \frac{1}{2}(\frac{2}{15}) = \frac{1}{15}$.
   - $\bar{x} = \frac{1/12}{1/6} = \frac{1}{2}$, $\bar{y} = \frac{1/15}{1/6} = \frac{2}{5}$.

   Answer: (\frac12, \frac25)

3. Find the centroid of the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

   - Area $A = \int_0^1 e^x\,dx = [e^x]_0^1 = e - 1$.
   - Moment $M_y = \int_0^1 xe^x\,dx$. Using integration by parts: $[xe^x - e^x]_0^1 = (e - e) - (0 - 1) = 1$.
   - Moment $M_x = \int_0^1 \frac{1}{2}(e^x)^2\,dx = \int_0^1 \frac{1}{2}e^{2x}\,dx = [\frac{1}{4}e^{2x}]_0^1 = \frac{1}{4}(e^2 - 1)$.
   - $\bar{x} = \frac{1}{e-1}$.
   - $\bar{y} = \frac{(e^2-1)/4}{e-1} = \frac{(e-1)(e+1)}{4(e-1)} = \frac{e+1}{4}$.

   Answer: (\frac1e-1, \frace+14)