

Integrals with symmetry — even/odd shortcuts and the king property

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Practice problems

1. Evaluate the definite integral $\int_{-\pi}^{\pi} x \sin(x) dx$ using symmetry.
2. Evaluate $\int_{-1}^1 (x^5 + x^3 + x) \cos(x) dx$.
3. Use the King Property to evaluate $\int_0^{\pi/4} \frac{1}{1+\tan(x)} dx$.

Solutions

1. Evaluate the definite integral $\int_{-\pi}^{\pi} x \sin(x) dx$ using symmetry.

- Check the symmetry of $f(x) = x \sin(x)$.
- Substitute $-x$: $f(-x) = (-x) \sin(-x)$.
- Since $\sin(-x) = -\sin(x)$, we have $f(-x) = (-x)(-\sin(x)) = x \sin(x)$.
- The function is even, so the integral is $2 \int_0^{\pi} x \sin(x) dx$.
- Use integration by parts: $\int x \sin(x) dx = -x \cos(x) + \sin(x)$.
- Evaluate from 0 to π : $[-\pi \cos(\pi) + \sin(\pi)] - [0 + 0] = [-\pi(-1) + 0] = \pi$.
- Multiply by 2 to get 2π .

Answer: 2π

2. Evaluate $\int_{-1}^1 (x^5 + x^3 + x) \cos(x) dx$.

- Let $f(x) = (x^5 + x^3 + x) \cos(x)$.
- Check symmetry: $f(-x) = ((-x)^5 + (-x)^3 + (-x)) \cos(-x)$.
- This simplifies to $(-x^5 - x^3 - x) \cos(x) = -(x^5 + x^3 + x) \cos(x)$.
- Since $f(-x) = -f(x)$, the function is odd.
- The integral of an odd function over a symmetric interval $[-1, 1]$ is 0.

Answer: 0

3. Use the King Property to evaluate $\int_0^{\pi/4} \frac{1}{1+\tan(x)} dx$.

- Let $I = \int_0^{\pi/4} \frac{1}{1+\tan(x)} dx$.
- Apply the King Property: replace x with $\pi/4 - x$.
- Use the identity $\tan(\pi/4 - x) = \frac{1-\tan(x)}{1+\tan(x)}$.
- The integrand becomes $\frac{1}{1+\frac{1-\tan(x)}{1+\tan(x)}} = \frac{1+\tan(x)}{(1+\tan(x))+(1-\tan(x))} = \frac{1+\tan(x)}{2}$.
- Add the two versions of I : $2I = \int_0^{\pi/4} \left(\frac{1}{1+\tan(x)} + \frac{1+\tan(x)}{2} \right) dx$.
- Alternatively, rewrite the integrand as $\frac{\cos(x)}{\cos(x)+\sin(x)}$.
- By King Property, $I = \int_0^{\pi/4} \frac{\sin(x)}{\sin(x)+\cos(x)} dx$.
- Then $2I = \int_0^{\pi/4} 1 dx = \pi/4$.
- Thus $I = \pi/8$.

Answer: $\pi/8$