

Improper integrals II — discontinuous integrands

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Practice problems

1. Evaluate the improper integral $\int_0^1 \frac{1}{x^{3/2}} dx$. Does it converge or diverge?
2. Determine if the integral $\int_0^1 \frac{1}{\sqrt{x+x^2}} dx$ converges or diverges using the comparison test.
3. Evaluate $\int_{-1}^1 \frac{1}{\sqrt[3]{x}} dx$.

Solutions

1. Evaluate the improper integral $\int_0^1 \frac{1}{x^{3/2}} dx$. Does it converge or diverge?

- Identify the discontinuity at $x = 0$.
- Rewrite as a limit: $\lim_{t \rightarrow 0^+} \int_t^1 x^{-3/2} dx$.
- Find the antiderivative: $\int x^{-3/2} dx = -2x^{-1/2} = -\frac{2}{\sqrt{x}}$.
- Evaluate the limit: $\lim_{t \rightarrow 0^+} [-\frac{2}{\sqrt{1}} - (-\frac{2}{\sqrt{t}})] = \lim_{t \rightarrow 0^+} (-2 + \frac{2}{\sqrt{t}})$.
- As $t \rightarrow 0^+$, the term $\frac{2}{\sqrt{t}}$ approaches infinity.

Answer: Diverges

2. Determine if the integral $\int_0^1 \frac{1}{\sqrt{x+x^2}} dx$ converges or diverges using the comparison test.

- Observe that for $x \in (0, 1]$, $\sqrt{x+x^2} > \sqrt{x}$.
- Therefore, $\frac{1}{\sqrt{x+x^2}} < \frac{1}{\sqrt{x}}$.
- The integral $\int_0^1 \frac{1}{\sqrt{x}} dx$ is a p-integral with $p = 1/2$.
- Since $p < 1$, the integral $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges.
- By the comparison test, since the original function is smaller than a convergent function, it also converges.

Answer: Converges

3. Evaluate $\int_{-1}^1 \frac{1}{\sqrt[3]{x}} dx$.

- Identify the discontinuity at $x = 0$. Split the integral: $\int_{-1}^0 x^{-1/3} dx + \int_0^1 x^{-1/3} dx$.
- Evaluate the right side: $\lim_{t \rightarrow 0^+} [\frac{3}{2}x^{2/3}]_t^1 = \frac{3}{2}(1)^{2/3} - 0 = \frac{3}{2}$.
- Evaluate the left side: $\lim_{t \rightarrow 0^-} [\frac{3}{2}x^{2/3}]_{-1}^t = 0 - \frac{3}{2}(-1)^{2/3} = -\frac{3}{2}$.
- Sum the two parts: $\frac{3}{2} + (-\frac{3}{2}) = 0$.

Answer: 0