

Improper integrals I — infinite limits

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Practice problems

1. Evaluate the improper integral $\int_1^\infty \frac{1}{x^3} dx$. Does it converge or diverge?
2. Determine if the integral $\int_1^\infty \frac{1}{x^2+1} dx$ converges or diverges. Find its value if it converges.
3. Evaluate $\int_0^\infty xe^{-x} dx$.

Solutions

1. Evaluate the improper integral $\int_1^\infty \frac{1}{x^3} dx$. Does it converge or diverge?

- Rewrite as a limit: $\lim_{R \rightarrow \infty} \int_1^R x^{-3} dx$.
- Find the antiderivative: $\int x^{-3} dx = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$.
- Evaluate from 1 to R : $\lim_{R \rightarrow \infty} [-\frac{1}{2R^2} - (-\frac{1}{2(1)^2})]$.
- Take the limit: $0 + \frac{1}{2} = \frac{1}{2}$.

Answer: Converges to $\frac{1}{2}$

2. Determine if the integral $\int_1^\infty \frac{1}{x^2+1} dx$ converges or diverges. Find its value if it converges.

- Rewrite as a limit: $\lim_{R \rightarrow \infty} \int_1^R \frac{1}{x^2+1} dx$.
- The antiderivative is $\arctan(x)$.
- Evaluate from 1 to R : $\lim_{R \rightarrow \infty} [\arctan(R) - \arctan(1)]$.
- The limit of $\arctan(R)$ as $R \rightarrow \infty$ is $\pi/2$.
- The value of $\arctan(1)$ is $\pi/4$.
- Result: $\pi/2 - \pi/4 = \pi/4$.

Answer: Converges to $\frac{\pi}{4}$

3. Evaluate $\int_0^\infty x e^{-x} dx$.

- Rewrite as $\lim_{R \rightarrow \infty} \int_0^R x e^{-x} dx$.
- Use integration by parts: $u = x, dv = e^{-x} dx \implies du = dx, v = -e^{-x}$.
- Integral is $[-x e^{-x}]_0^R - \int_0^R -e^{-x} dx = [-x e^{-x} - e^{-x}]_0^R$.
- Evaluate: $\lim_{R \rightarrow \infty} (-R e^{-R} - e^{-R}) - (0 - e^0)$.
- Since $R e^{-R} \rightarrow 0$ and $e^{-R} \rightarrow 0$ as $R \rightarrow \infty$, we are left with $0 - (-1) = 1$.

Answer: Converges to 1