

Mixed integration gauntlet II

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Practice problems

1. Find the indefinite integral $\int x(\ln x)^2 dx$.
2. Evaluate the integral $\int \frac{1}{1+\cos x} dx$.
3. Find the antiderivative of $\int e^x \sin(2x) dx$.

Solutions

1. Find the indefinite integral $\int x(\ln x)^2 dx$.

- Use integration by parts with $u = (\ln x)^2$ and $dv = x dx$.
- Then $du = 2 \ln x \cdot \frac{1}{x} dx$ and $v = \frac{1}{2}x^2$.
- The formula gives $\frac{1}{2}x^2(\ln x)^2 - \int \frac{1}{2}x^2 \cdot \frac{2 \ln x}{x} dx$, which simplifies to $\frac{1}{2}x^2(\ln x)^2 - \int x \ln x dx$.
- Apply integration by parts again to $\int x \ln x dx$ with $u = \ln x$ and $dv = x dx$.
- This gives $\frac{1}{2}x^2 \ln x - \int \frac{1}{2}x dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$.
- Combine all terms: $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$.

Answer: $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$

2. Evaluate the integral $\int \frac{1}{1+\cos x} dx$.

- Use the Weierstrass substitution $t = \tan(x/2)$.
- Substitute $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2} dt$.
- The integral becomes $\int \frac{1}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt$.
- Simplify the denominator: $1 + \frac{1-t^2}{1+t^2} = \frac{1+t^2+1-t^2}{1+t^2} = \frac{2}{1+t^2}$.
- The integral simplifies to $\int \frac{1+t^2}{2} \cdot \frac{2}{1+t^2} dt = \int 1 dt = t + C$.
- Substitute back $t = \tan(x/2)$.

Answer: $\tan(x/2) + C$

3. Find the antiderivative of $\int e^x \sin(2x) dx$.

- Use integration by parts with $u = \sin(2x)$ and $dv = e^x dx$.
- Then $du = 2 \cos(2x) dx$ and $v = e^x$.
- The first pass gives $e^x \sin(2x) - \int 2e^x \cos(2x) dx$.
- Apply integration by parts again to $\int 2e^x \cos(2x) dx$ with $u = 2 \cos(2x)$ and $dv = e^x dx$.
- Then $du = -4 \sin(2x) dx$ and $v = e^x$.
- This gives $2e^x \cos(2x) - \int -4e^x \sin(2x) dx = 2e^x \cos(2x) + 4 \int e^x \sin(2x) dx$.
- Let $I = \int e^x \sin(2x) dx$. We have $I = e^x \sin(2x) - (2e^x \cos(2x) + 4I)$.
- Solve for I : $5I = e^x \sin(2x) - 2e^x \cos(2x) \implies I = \frac{e^x}{5}(\sin(2x) - 2 \cos(2x)) + C$.

Answer: $\frac{e^x}{5}(\sin(2x) - 2 \cos(2x)) + C$