Rationalising substitutions and Weierstrass t = tan(x/2)

Practice problems

1. Evaluate the integral $\int \frac{1}{1-\cos x}\,dx$.
2. Evaluate the integral $\int \frac{1}{2+\cos x}\,dx$.
3. Evaluate the integral $\int \frac{\sin x}{1+\sin x+\cos x}\,dx$.

Solutions

1. Evaluate the integral $\int \frac{1}{1-\cos x}\,dx$.

   - Substitute $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$.
   - The denominator becomes $1 - \frac{1-t^2}{1+t^2} = \frac{1+t^2-(1-t^2)}{1+t^2} = \frac{2t^2}{1+t^2}$.
   - The integral is $\int \frac{1}{2t^2/(1+t^2)} \cdot \frac{2}{1+t^2}\,dt = \int \frac{1}{t^2}\,dt$.
   - Integrating t^{-2} gives $-t^{-1} + C = -\frac{1}{t} + C$.
   - Substitute back $t = \tan(x/2)$ to get $-\cot(x/2) + C$.

   Answer: -\cot\left(\fracx2\right) + C

2. Evaluate the integral $\int \frac{1}{2+\cos x}\,dx$.

   - Substitute $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$.
   - The denominator is $2 + \frac{1-t^2}{1+t^2} = \frac{2(1+t^2)+1-t^2}{1+t^2} = \frac{3+t^2}{1+t^2}$.
   - The integral becomes $\int \frac{1+t^2}{3+t^2} \cdot \frac{2}{1+t^2}\,dt = \int \frac{2}{3+t^2}\,dt$.
   - This is an arctangent form: $2 \cdot \frac{1}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) + C$.
   - Substitute back $t = \tan(x/2)$.

   Answer: \frac2\sqrt3 \arctan\left(\frac\tan(x/2)\sqrt3\right) + C

3. Evaluate the integral $\int \frac{\sin x}{1+\sin x+\cos x}\,dx$.

   - Substitute $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, and $dx = \frac{2}{1+t^2}dt$.
   - The integral becomes $\int \frac{2t/(1+t^2)}{1+(2t+1-t^2)/(1+t^2)} \cdot \frac{2}{1+t^2}\,dt$.
   - Simplify the denominator: $1 + \frac{2t+1-t^2}{1+t^2} = \frac{1+t^2+2t+1-t^2}{1+t^2} = \frac{2+2t}{1+t^2}$.
   - The integral is $\int \frac{2t}{1+t^2} \cdot \frac{1+t^2}{2+2t} \cdot \frac{2}{1+t^2}\,dt = \int \frac{2t}{(1+t)(1+t^2)}\,dt$.
   - Use partial fractions: $\frac{2t}{(1+t)(1+t^2)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2}$. Solving gives $A = -1, B = 1, C = 1$.
   - Integrate: $\int (\frac{-1}{1+t} + \frac{t}{1+t^2} + \frac{1}{1+t^2})\,dt = -\ln|1+t| + \frac{1}{2}\ln(1+t^2) + \arctan(t) + C$.

   Answer: -\ln—1 + \tan(x/2)— + \frac12\ln(1 + \tanˆ2(x/2)) + \fracx2 + C