

Trigonometric substitution II

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Practice problems

1. Evaluate the integral $\int \frac{1}{x^2+25} dx$.
2. Evaluate the integral $\int \frac{1}{\sqrt{x^2-9}} dx$.
3. Evaluate the integral $\int \frac{x^3}{\sqrt{x^2+4}} dx$.

Solutions

1. Evaluate the integral $\int \frac{1}{x^2+25} dx$.

- Identify the form $\sqrt{a^2 + x^2}$ with $a = 5$.
- Substitute $x = 5 \tan \theta$, so $dx = 5 \sec^2 \theta d\theta$.
- The denominator becomes $25 \tan^2 \theta + 25 = 25 \sec^2 \theta$.
- The integral becomes $\int \frac{5 \sec^2 \theta}{25 \sec^2 \theta} d\theta = \int \frac{1}{5} d\theta = \frac{1}{5} \theta + C$.
- Since $\tan \theta = \frac{x}{5}$, then $\theta = \arctan(\frac{x}{5})$.

Answer: $\frac{1}{5} \arctan\left(\frac{x}{5}\right) + C$

2. Evaluate the integral $\int \frac{1}{\sqrt{x^2-9}} dx$.

- Identify the form $\sqrt{x^2 - a^2}$ with $a = 3$.
- Substitute $x = 3 \sec \theta$, so $dx = 3 \sec \theta \tan \theta d\theta$.
- The square root becomes $\sqrt{9 \sec^2 \theta - 9} = 3 \tan \theta$.
- The integral becomes $\int \frac{3 \sec \theta \tan \theta}{3 \tan \theta} d\theta = \int \sec \theta d\theta$.
- The antiderivative of $\sec \theta$ is $\ln |\sec \theta + \tan \theta| + C$.
- Substitute back $\sec \theta = \frac{x}{3}$ and $\tan \theta = \frac{\sqrt{x^2-9}}{3}$, giving $\ln \left| \frac{x+\sqrt{x^2-9}}{3} \right| + C$.

Answer: $\ln \left| \frac{x+\sqrt{x^2-9}}{3} \right| + C$

3. Evaluate the integral $\int \frac{x^3}{\sqrt{x^2+4}} dx$.

- Substitute $x = 2 \tan \theta$, so $dx = 2 \sec^2 \theta d\theta$.
- The square root is $2 \sec \theta$ and $x^3 = 8 \tan^3 \theta$.
- The integral becomes $\int \frac{8 \tan^3 \theta}{2 \sec \theta} (2 \sec^2 \theta) d\theta = 8 \int \tan^3 \theta \sec \theta d\theta$.
- Rewrite as $8 \int \tan \theta (\tan^2 \theta \sec \theta) d\theta = 8 \int \tan \theta (\sec^2 \theta - 1) \sec \theta d\theta$.
- Alternatively, use $u = \sec \theta$, then $du = \sec \theta \tan \theta d\theta$.
- The integral is $8 \int (u^2 - 1) du = 8 \left(\frac{u^3}{3} - u \right) + C = 8 \left(\frac{\sec^3 \theta}{3} - \sec \theta \right) + C$.
- Substitute back $\sec \theta = \frac{\sqrt{x^2+4}}{2}$.

Answer: $\frac{1}{3} (x^2+4)^{3/2} - 4 \sqrt{x^2+4} + C$