

Trigonometric substitution I

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Practice problems

1. Evaluate the indefinite integral $\int \frac{1}{\sqrt{25-x^2}} dx$.
2. Evaluate the indefinite integral $\int \frac{1}{(4-x^2)^{3/2}} dx$.
3. Evaluate the indefinite integral $\int \sqrt{9-x^2} dx$.

Solutions

1. Evaluate the indefinite integral $\int \frac{1}{\sqrt{25-x^2}} dx$.
- Identify $a = 5$ and use the substitution $x = 5 \sin \theta$.
 - Compute the differential $dx = 5 \cos \theta d\theta$.
 - Substitute into the integral: $\int \frac{5 \cos \theta}{\sqrt{25-25 \sin^2 \theta}} d\theta = \int \frac{5 \cos \theta}{5 \cos \theta} d\theta$.
 - Integrate $1 d\theta$ to get $\theta + C$.
 - Back-substitute using $\sin \theta = x/5$, so $\theta = \arcsin(x/5)$.

Answer: $\arcsin\left(\frac{x}{5}\right) + C$

2. Evaluate the indefinite integral $\int \frac{1}{(4-x^2)^{3/2}} dx$.
- Use $x = 2 \sin \theta$, so $dx = 2 \cos \theta d\theta$.
 - The term $(4-x^2)^{3/2}$ becomes $(4 \cos^2 \theta)^{3/2} = 8 \cos^3 \theta$.
 - The integral becomes $\int \frac{2 \cos \theta}{8 \cos^3 \theta} d\theta = \frac{1}{4} \int \sec^2 \theta d\theta$.
 - Integrate to get $\frac{1}{4} \tan \theta + C$.
 - Using the triangle, $\tan \theta = \frac{x}{\sqrt{4-x^2}}$.

Answer: $\frac{x}{4\sqrt{4-x^2}} + C$

3. Evaluate the indefinite integral $\int \sqrt{9-x^2} dx$.
- Use $x = 3 \sin \theta$, so $dx = 3 \cos \theta d\theta$.
 - The integral becomes $\int (3 \cos \theta)(3 \cos \theta) d\theta = 9 \int \cos^2 \theta d\theta$.
 - Use the identity $\cos^2 \theta = \frac{1+\cos(2\theta)}{2}$ to get $\frac{9}{2} \int (1 + \cos(2\theta)) d\theta$.
 - Integrate to get $\frac{9}{2} \theta + \frac{9}{4} \sin(2\theta) + C$.
 - Convert back: $\theta = \arcsin(x/3)$ and $\sin(2\theta) = 2(x/3)(\sqrt{9-x^2}/3) = \frac{2x\sqrt{9-x^2}}{9}$.

Answer: $\frac{9}{2} \arcsin\left(\frac{x}{3}\right) + \frac{1}{2} x \sqrt{9-x^2} + C$