

Trigonometric integrals (powers of sin and cos)

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Practice problems

1. Find the indefinite integral $\int \sin^5 x \, dx$.
2. Evaluate the indefinite integral $\int \sin^3 x \cos^3 x \, dx$.
3. Find the indefinite integral $\int \cos^4 x \, dx$.

Solutions

1. Find the indefinite integral $\int \sin^5 x \, dx$.

- Since the power is odd, peel off one $\sin x$ factor: $\int (\sin^2 x)^2 \sin x \, dx$.
- Use the identity $\sin^2 x = 1 - \cos^2 x$ to get $\int (1 - \cos^2 x)^2 \sin x \, dx$.
- Let $u = \cos x$, then $du = -\sin x \, dx$. The integral becomes $-\int (1 - u^2)^2 \, du$.
- Expand the binomial: $-\int (1 - 2u^2 + u^4) \, du$.
- Integrate term by term: $-(u - \frac{2}{3}u^3 + \frac{1}{5}u^5) + C$.
- Substitute back $u = \cos x$: $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$.

Answer: $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

2. Evaluate the indefinite integral $\int \sin^3 x \cos^3 x \, dx$.

- Both powers are odd, so we can choose either. Let's peel one $\cos x$ and let $u = \sin x$.
- The integral becomes $\int \sin^3 x (\cos^2 x) \cos x \, dx = \int \sin^3 x (1 - \sin^2 x) \cos x \, dx$.
- Substitute $u = \sin x$ and $du = \cos x \, dx$: $\int u^3 (1 - u^2) \, du$.
- Distribute: $\int (u^3 - u^5) \, du$.
- Integrate: $\frac{1}{4}u^4 - \frac{1}{6}u^6 + C$.
- Substitute back: $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$.

Answer: $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

3. Find the indefinite integral $\int \cos^4 x \, dx$.

- The power is even, so use the half-angle identity: $\int (\frac{1+\cos(2x)}{2})^2 \, dx$.
- Expand the square: $\int \frac{1+2\cos(2x)+\cos^2(2x)}{4} \, dx$.
- Split the integral: $\frac{1}{4} \int 1 \, dx + \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \cos^2(2x) \, dx$.
- Apply the half-angle identity again to $\cos^2(2x)$: $\int \frac{1+\cos(4x)}{2} \, dx$.
- Integrate all parts: $\frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{4}(\frac{1}{2}x + \frac{1}{8}\sin(4x)) + C$.
- Simplify: $\frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$.

Answer: $\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$