Integration by parts II — repeated and cyclic cases

Practice problems

1. Evaluate the indefinite integral $\int x^3 e^x \, dx$.
2. Evaluate the indefinite integral $\int e^{2x} \cos(x) \, dx$.
3. Evaluate the indefinite integral $\int x^2 \sin(x) \, dx$.

Solutions

1. Evaluate the indefinite integral $\int x^3 e^x \, dx$.

   - Use integration by parts with $u = x^3$ and $dv = e^x dx$. This gives $x^3 e^x - \int 3x^2 e^x dx$.
   - Apply integration by parts again to $\int 3x^2 e^x dx$ with $u = 3x^2$ and $dv = e^x dx$. This gives $3x^2 e^x - \int 6x e^x dx$.
   - Apply integration by parts a third time to $\int 6x e^x dx$ with $u = 6x$ and $dv = e^x dx$. This gives $6x e^x - \int 6e^x dx$.
   - The final integral is $6e^x$. Combine all terms: $x^3 e^x - (3x^2 e^x - (6x e^x - 6e^x))$.
   - Simplify the signs: $x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$.

   Answer: (xˆ3 - 3xˆ2 + 6x - 6)eˆx + C

2. Evaluate the indefinite integral $\int e^{2x} \cos(x) \, dx$.

   - Let $I = \int e^{2x} \cos x dx$. Set $u = \cos x$ and $dv = e^{2x} dx$. Then $du = -\sin x dx$ and $v = \frac{1}{2} e^{2x}$.
   - First pass: $I = \frac{1}{2} e^{2x} \cos x + \int \frac{1}{2} e^{2x} \sin x dx$.
   - Second pass on $\int \frac{1}{2} e^{2x} \sin x dx$: set $u = \sin x$ and $dv = \frac{1}{2} e^{2x} dx$. Then $du = \cos x dx$ and $v = \frac{1}{4} e^{2x}$.
   - The second integral is $\frac{1}{4} e^{2x} \sin x - \int \frac{1}{4} e^{2x} \cos x dx$.
   - Substitute back: $I = \frac{1}{2} e^{2x} \cos x + \frac{1}{4} e^{2x} \sin x - \frac{1}{4} I$.
   - Add $\frac{1}{4} I$ to both sides: $\frac{5}{4} I = e^{2x} (\frac{1}{2} \cos x + \frac{1}{4} \sin x)$.
   - Multiply by $\frac{4}{5}$: $I = \frac{4}{5} e^{2x} (\frac{1}{2} \cos x + \frac{1}{4} \sin x) = \frac{e^{2x}}{5} (2 \cos x + \sin x) + C$.

   Answer: \fracˆ2x5 (2 \cos x + \sin x) + C

3. Evaluate the indefinite integral $\int x^2 \sin(x) \, dx$.

   - Set $u = x^2$ and $dv = \sin x dx$. Then $du = 2x dx$ and $v = -\cos x$.
   - First pass: $-x^2 \cos x + \int 2x \cos x dx$.
   - Second pass on $\int 2x \cos x dx$: set $u = 2x$ and $dv = \cos x dx$. Then $du = 2dx$ and $v = \sin x$.
   - The second integral is $2x \sin x - \int 2 \sin x dx$.
   - The final integral is $-2(-\cos x) = 2 \cos x$.
   - Combine: $-x^2 \cos x + 2x \sin x + 2 \cos x + C$.

   Answer: (2 - xˆ2)\cos x + 2x \sin x + C