

Optimisation II — harder problems + module review

Lesson 56 · Module 2 — Differential Calculus · The Calculus Course · dailymaths.uk/course

Practice problems

1. A rectangular box with a square base and an open top must have a volume of 10 cubic meters. Find the dimensions that minimize the surface area.
2. A piece of wire of length 20 cm is cut into two pieces. One piece is bent into a square and the other into a circle. Where should the cut be made to minimize the total area enclosed?
3. A light ray travels from a medium where its speed is v_1 to a medium where its speed is v_2 . If the angle of incidence is 30° and the angle of refraction is 45° , find the ratio v_1/v_2 .

Solutions

1. A rectangular box with a square base and an open top must have a volume of 10 cubic meters. Find the dimensions that minimize the surface area.

- Let x be the side of the square base and h be the height. The volume is $V = x^2h = 10$, so $h = 10/x^2$.
- The surface area is $A = x^2 + 4xh$. Substituting h , we get $A(x) = x^2 + 4x(10/x^2) = x^2 + 40/x$.
- Find the derivative: $A'(x) = 2x - 40/x^2$.
- Set $A'(x) = 0$: $2x = 40/x^2$, which means $x^3 = 20$, so $x = \sqrt[3]{20}$.
- Find h : $h = 10/(\sqrt[3]{20})^2 = 10/20^{2/3} = (10 \cdot 20^{1/3})/20 = 20^{1/3}/2 = \sqrt[3]{20}/2$.

Answer: $x = \sqrt[3]{20}$, $h = \sqrt[3]{20}/2$

2. A piece of wire of length 20 cm is cut into two pieces. One piece is bent into a square and the other into a circle. Where should the cut be made to minimize the total area enclosed?

- Let x be the length of the wire used for the square. The remaining $20 - x$ is used for the circle.
- Side of square is $x/4$, area is $(x/4)^2 = x^2/16$.
- Circumference of circle is $2\pi r = 20 - x$, so $r = (20 - x)/(2\pi)$, area is $\pi((20 - x)/(2\pi))^2 = (20 - x)^2/(4\pi)$.
- Total area $A(x) = x^2/16 + (20 - x)^2/(4\pi)$.
- Derivative $A'(x) = x/8 - (20 - x)/(2\pi)$.
- Set $A'(x) = 0$: $x/8 = (20 - x)/(2\pi) \implies \pi x = 4(20 - x) \implies x(\pi + 4) = 80 \implies x = 80/(\pi + 4)$.

Answer: $x = \frac{80}{\pi + 4}$

3. A light ray travels from a medium where its speed is v_1 to a medium where its speed is v_2 . If the angle of incidence is 30° and the angle of refraction is 45° , find the ratio v_1/v_2 .

- Use Snell's law: $\sin(\theta_1)/v_1 = \sin(\theta_2)/v_2$.
- Rearrange for the ratio: $v_1/v_2 = \sin(\theta_1)/\sin(\theta_2)$.
- Substitute the angles: $v_1/v_2 = \sin(30^\circ)/\sin(45^\circ)$.
- Calculate values: $\sin(30^\circ) = 1/2$ and $\sin(45^\circ) = \sqrt{2}/2$.
- Ratio: $(1/2)/(\sqrt{2}/2) = 1/\sqrt{2} = \sqrt{2}/2$.

Answer: $\frac{\sqrt{2}}{2}$