

Curve sketching masterclass II

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Practice problems

1. Sketch the function $f(x) = x^2e^{-x}$ for $x \geq 0$. Find all critical points and inflection points.
2. Sketch $f(x) = e^x \cos(x)$ on the interval $[0, 2\pi]$.
3. Analyze and sketch $f(x) = x \cos(x)$ on the interval $[0, \pi]$.

Solutions

1. Sketch the function $f(x) = x^2e^{-x}$ for $x \geq 0$. Find all critical points and inflection points.

- Find $f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$.
- Set $f'(x) = 0$ to find critical points at $x = 0$ and $x = 2$.
- Find $f''(x) = (2 - 2x)e^{-x} - (2x - x^2)e^{-x} = e^{-x}(x^2 - 4x + 2)$.
- Solve $x^2 - 4x + 2 = 0$ using the quadratic formula to find inflection points at $x = 2 \pm \sqrt{2}$.
- Analyze signs: f increases on $(0, 2)$ and decreases on $(2, \infty)$.
- Plot the local maximum at $(2, 4e^{-2})$ and the inflection points.

Answer: Local max at $x = 2$, inflection points at $x = 2 - \sqrt{2}$ and $x = 2 + \sqrt{2}$, horizontal asymptote $y = 0$ as $x \rightarrow \infty$.

2. Sketch $f(x) = e^x \cos(x)$ on the interval $[0, 2\pi]$.

- Find $f'(x) = e^x \cos(x) - e^x \sin(x) = e^x(\cos(x) - \sin(x))$.
- Set $f'(x) = 0 \implies \tan(x) = 1$, so $x = \pi/4$ and $x = 5\pi/4$.
- Find $f''(x) = e^x(\cos(x) - \sin(x)) + e^x(-\sin(x) - \cos(x)) = -2e^x \sin(x)$.
- Set $f''(x) = 0 \implies \sin(x) = 0$, so $x = 0, \pi, 2\pi$.
- Determine concavity: concave down on $(0, \pi)$ and concave up on $(\pi, 2\pi)$.

Answer: Local max at $x = \pi/4$, local min at $x = 5\pi/4$, inflection point at $x = \pi$.

3. Analyze and sketch $f(x) = x \cos(x)$ on the interval $[0, \pi]$.

- Find $f'(x) = \cos(x) - x \sin(x)$.
- Set $f'(x) = 0 \implies \cot(x) = x$. This occurs at $x \approx 0.86$ radians.
- Find $f''(x) = -\sin(x) - (\sin(x) + x \cos(x)) = -2\sin(x) - x \cos(x)$.
- Set $f''(x) = 0 \implies \tan(x) = -x/2$. This occurs at $x \approx 2.28$ radians.
- Check endpoints: $f(0) = 0$ and $f(\pi) = -\pi$.

Answer: Local max at $x \approx 0.86$, inflection point at $x \approx 2.28$, decreasing from the max to the endpoint.