

Curve sketching masterclass I

Lesson 53 · Module 2 — Differential Calculus · The Calculus Course · dailymaths.uk/course

Practice problems

1. Perform a full curve sketching analysis for $f(x) = \frac{x+1}{x-2}$. Find the domain, asymptotes, first and second derivatives, and sketch the graph.
2. Analyze and sketch $f(x) = \frac{1}{x^2+1}$. Identify all extrema and inflection points.
3. Sketch the function $f(x) = \frac{x^2-1}{x^2-4}$. Find the vertical asymptotes, the horizontal asymptote, and the local extrema.

Solutions

1. Perform a full curve sketching analysis for $f(x) = \frac{x+1}{x-2}$. Find the domain, asymptotes, first and second derivatives, and sketch the graph.

- Domain is $x \neq 2$. Vertical asymptote at $x = 2$.
- Limit as $x \rightarrow \infty$ is 1, so horizontal asymptote is $y = 1$.
- Using quotient rule, $f'(x) = \frac{1(x-2)-1(x+1)}{(x-2)^2} = \frac{-3}{(x-2)^2}$. Since $f'(x) < 0$ everywhere, the function is always decreasing.
- Using quotient rule on $f'(x)$, $f''(x) = \frac{6}{(x-2)^3}$. Concave down for $x < 2$ and concave up for $x > 2$.
- Intercepts: y -intercept at $f(0) = -1/2$, x -intercept at $f(x) = 0 \implies x = -1$.

Answer: A hyperbola with asymptotes $x = 2, y = 1$, always decreasing, concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

2. Analyze and sketch $f(x) = \frac{1}{x^2+1}$. Identify all extrema and inflection points.

- Domain is all real numbers. No vertical asymptotes. Horizontal asymptote at $y = 0$.
- First derivative $f'(x) = \frac{-2x}{(x^2+1)^2}$. Critical point at $x = 0$. $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$. Local max at $(0, 1)$.
- Second derivative $f''(x) = \frac{2(3x^2-1)}{(x^2+1)^3}$. Inflection points where $3x^2 = 1$, so $x = \pm 1/\sqrt{3}$.
- Concave up for $|x| > 1/\sqrt{3}$ and concave down for $|x| < 1/\sqrt{3}$.

Answer: A bell-shaped curve with a maximum at $(0, 1)$, horizontal asymptote $y = 0$, and inflection points at $x = \pm 1/\sqrt{3}$.

3. Sketch the function $f(x) = \frac{x^2-1}{x^2-4}$. Find the vertical asymptotes, the horizontal asymptote, and the local extrema.

- Domain is $x \neq \pm 2$. Vertical asymptotes at $x = -2$ and $x = 2$.
- Limit as $x \rightarrow \infty$ is 1, so horizontal asymptote is $y = 1$.
- First derivative $f'(x) = \frac{2x(x^2-4)-2x(x^2-1)}{(x^2-4)^2} = \frac{-6x}{(x^2-4)^2}$.
- Critical point at $x = 0$. $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$. Local max at $(0, 1/4)$.
- Intercepts: y -intercept at $(0, 1/4)$, x -intercepts at $x = \pm 1$.

Answer: A graph with vertical asymptotes $x = \pm 2$, horizontal asymptote $y = 1$, and a local maximum at $(0, 1/4)$.