Concavity, inflection points, second derivative test

Practice problems

1. Find the intervals of concavity and the x-coordinates of any inflection points for $f(x) = x^4 - 6x^2 + 5$.
2. Use the second derivative test to classify the critical points of $f(x) = xe^x$.
3. Determine if $f(x) = x^5$ has a local maximum, local minimum, or inflection point at $x = 0$.

Solutions

1. Find the intervals of concavity and the x-coordinates of any inflection points for $f(x) = x^4 - 6x^2 + 5$.

   - Find the first derivative: $f'(x) = 4x^3 - 12x$.
   - Find the second derivative: $f''(x) = 12x^2 - 12$.
   - Set $f''(x) = 0$ to find potential inflection points: $12(x^2 - 1) = 0$, so $x = 1$ and $x = -1$.
   - Test intervals: for $x < -1$, $f''(-2) = 36 > 0$ (Up); for $-1 < x < 1$, $f''(0) = -12 < 0$ (Down); for $x > 1$, $f''(2) = 36 > 0$ (Up).
   - Since the sign changes at both $x = -1$ and $x = 1$, both are inflection points.

   Answer: Concave up on $(-\infty, -1) \cup (1, \infty)$, concave down on $(-1, 1)$, inflection points at $x = -1, 1$.

2. Use the second derivative test to classify the critical points of $f(x) = xe^x$.

   - Find the first derivative using the product rule: $f'(x) = 1 \cdot e^x + x \cdot e^x = e^x(1 + x)$.
   - Set $f'(x) = 0$ to find critical points: $e^x(1 + x) = 0$ implies $x = -1$.
   - Find the second derivative: $f''(x) = e^x(1 + x) + e^x(1) = e^x(x + 2)$.
   - Evaluate at the critical point: $f''(-1) = e^{-1}(-1 + 2) = e^{-1} \approx 0.368$.
   - Since $f''(-1) > 0$, the function is concave up at $x = -1$.

   Answer: Local minimum at $x = -1$.

3. Determine if $f(x) = x^5$ has a local maximum, local minimum, or inflection point at $x = 0$.

   - Find the first derivative: $f'(x) = 5x^4$. Setting $f'(0) = 0$ confirms $x = 0$ is a critical point.
   - Find the second derivative: $f''(x) = 20x^3$.
   - Evaluate at $x = 0$: $f''(0) = 0$. The second derivative test is inconclusive.
   - Use the first derivative test: for $x < 0$, $f'(x) = 5x^4 > 0$; for $x > 0$, $f'(x) = 5x^4 > 0$.
   - Since the first derivative does not change sign, it is not a local extremum. Check concavity: for $x < 0$, $f''(x) = 20x^3 < 0$ (Down); for $x > 0$, $f''(x) = 20x^3 > 0$ (Up).
   - Since concavity changes, it is an inflection point.

   Answer: Inflection point at $x = 0$.