Increasing/decreasing functions; first derivative test

Lesson 51 · Module 2 — Differential Calculus · The Calculus Course · dailymaths.uk/course

Practice problems

1. Find the local maximum and minimum values of $f(x) = x^3 - 3x^2 - 9x + 5$.
2. Determine the intervals of increase and decrease for $f(x) = x^4 - 4x^2$.
3. Find the critical points of $f(x) = x^{2/3}$ and use the first derivative test to determine if they are local extrema.

Solutions

1. Find the local maximum and minimum values of $f(x) = x^3 - 3x^2 - 9x + 5$.

   - Find the derivative: $f'(x) = 3x^2 - 6x - 9$.
   - Set $f'(x) = 0$: $3(x^2 - 2x - 3) = 0$, which factors to $3(x-3)(x+1) = 0$. Critical points are $x = -1$ and $x = 3$.
   - Test intervals: For $x < -1$, $f'(-2) = 15 > 0$ (increasing). For $-1 < x < 3$, $f'(0) = -9 < 0$ (decreasing). For $x > 3$, $f'(4) = 15 > 0$ (increasing).
   - Classify: At $x = -1$, sign changes $(+) \to (-)$, so it is a local max. At $x = 3$, sign changes $(-) \to (+)$, so it is a local min.
   - Calculate values: $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$. $f(3) = (3)^3 - 3(3)^2 - 9(3) + 5 = -22$.

   Answer: Local maximum of 10 at $x = -1$; local minimum of -22 at $x = 3$.

2. Determine the intervals of increase and decrease for $f(x) = x^4 - 4x^2$.

   - Find the derivative: $f'(x) = 4x^3 - 8x$.
   - Set $f'(x) = 0$: $4x(x^2 - 2) = 0$. Critical points are $x = 0, x = \sqrt{2}, x = -\sqrt{2}$.
   - Test intervals: $x < -\sqrt{2}$ (e.g., -2): $f'(-2) = -32 + 16 = -16 < 0$ (decreasing).
   - Interval $-\sqrt{2} < x < 0$ (e.g., -1): $f'(-1) = -4 + 8 = 4 > 0$ (increasing).
   - Interval $0 < x < \sqrt{2}$ (e.g., 1): $f'(1) = 4 - 8 = -4 < 0$ (decreasing).
   - Interval $x > \sqrt{2}$ (e.g., 2): $f'(2) = 32 - 16 = 16 > 0$ (increasing).

   Answer: Increasing on $(-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$; decreasing on $(-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$.

3. Find the critical points of $f(x) = x^{2/3}$ and use the first derivative test to determine if they are local extrema.

   - Find the derivative: $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$.
   - Identify critical points: $f'(x)$ is never zero, but it is undefined at $x = 0$. Since $x = 0$ is in the domain of f, it is a critical point.
   - Test intervals: For $x < 0$, $\sqrt[3]{x}$ is negative, so $f'(x) < 0$ (decreasing).
   - For $x > 0$, $\sqrt[3]{x}$ is positive, so $f'(x) > 0$ (increasing).
   - Classify: The sign changes from negative to positive at $x = 0$, so it is a local minimum.

   Answer: Critical point at $x = 0$, which is a local minimum.
