Rolle's theorem and the Mean Value Theorem

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Practice problems

1. Find all values of c that satisfy the Mean Value Theorem for $f(x) = x^3 - x$ on the interval $[0, 2]$.
2. Does Rolle's theorem apply to $f(x) = x^{2/3}$ on the interval $[-1, 1]$? If so, find c. If not, explain why.
3. For $f(x) = \ln(x)$ on the interval $[1, e]$, find the value of c guaranteed by the Mean Value Theorem.

Solutions

1. Find all values of c that satisfy the Mean Value Theorem for $f(x) = x^3 - x$ on the interval $[0, 2]$.

   - Calculate the average slope: $f(0) = 0$ and $f(2) = 2^3 - 2 = 6$. The average slope is $(6 - 0)/(2 - 0) = 3$.
   - Find the derivative: $f'(x) = 3x^2 - 1$.
   - Set the derivative equal to the average slope: $3c^2 - 1 = 3$.
   - Solve for c: $3c^2 = 4$, so $c^2 = 4/3$, which means $c = \pm 2/\sqrt{3}$.
   - Check which value is in the interval $[0, 2]$. Only $c = 2/\sqrt{3}$ is valid.

   Answer: $c = \frac{2}{\sqrt{3}}$

2. Does Rolle's theorem apply to $f(x) = x^{2/3}$ on the interval $[-1, 1]$? If so, find c. If not, explain why.

   - Check continuity: $f(x)$ is continuous on $[-1, 1]$.
   - Check endpoints: $f(-1) = (-1)^{2/3} = 1$ and $f(1) = 1^{2/3} = 1$. Endpoints are equal.
   - Check differentiability: $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$.
   - The derivative is undefined at $x = 0$, which is inside the open interval $(-1, 1)$.
   - Since the function is not differentiable on the open interval, Rolle's theorem does not apply.

   Answer: No, because $f(x)$ is not differentiable at $x = 0$.

3. For $f(x) = \ln(x)$ on the interval $[1, e]$, find the value of c guaranteed by the Mean Value Theorem.

   - Calculate the average slope: $f(1) = \ln(1) = 0$ and $f(e) = \ln(e) = 1$. The average slope is $(1 - 0)/(e - 1) = 1/(e - 1)$.
   - Find the derivative: $f'(x) = 1/x$.
   - Set the derivative equal to the average slope: $1/c = 1/(e - 1)$.
   - Solve for c: $c = e - 1$.
   - Check if c is in the interval $[1, e]$. Since $e \approx 2.718$, $c \approx 1.718$, which is between 1 and 2.718.

   Answer: $c = e - 1$
