# L'Hopital's rule II — infinity forms and disguises

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## Practice problems

1. Evaluate the limit: $\lim_{x\to 0^+} x^2 \ln x$.
2. Evaluate the limit: $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$.
3. Evaluate the limit: $\lim_{x\to 0^+}(1+x)^{1/x}$.

## Solutions

1. Evaluate the limit: $\lim_{x\to 0^+} x^2 \ln x$.

   - Identify the form: as $x \to 0^+$, $x^2 \to 0$ and $\ln x \to -\infty$. This is a $0 \cdot \infty$ form.
   - Rewrite as a quotient: $\lim_{x\to 0^+} \frac{\ln x}{1/x^2}$. This is now $-\infty/\infty$.
   - Apply L'Hopital's rule: the derivative of $\ln x$ is $1/x$ and the derivative of $1/x^2$ is $-2/x^3$.
   - Simplify the expression: $\frac{1/x}{-2/x^3} = \frac{1}{x} \cdot \frac{x^3}{-2} = -\frac{x^2}{2}$.
   - Evaluate the limit: as $x \to 0^+$, $-x^2/2 \to 0$.

   **Answer:** 0

2. Evaluate the limit: $\lim_{x\to\infty}(\sqrt{x^2+x} - x)$.

   - Identify the form: as $x \to \infty$, both terms go to infinity, so this is $\infty - \infty$.
   - Rationalize the expression by multiplying by the conjugate: $\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}$.
   - Simplify the numerator: $(\sqrt{x^2+x})^2 - x^2 = x^2 + x - x^2 = x$.
   - The limit is now $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x}$, which is $\infty/\infty$.
   - Apply L'Hopital's rule: the derivative of the top is 1. The derivative of the bottom is $\frac{2x+1}{2\sqrt{x^2+x}} + 1$.
   - As $x \to \infty$, the fraction $\frac{2x+1}{2\sqrt{x^2+x}}$ approaches 1.
   - The limit becomes $1/(1+1) = 1/2$.

   **Answer:** $1/2$

3. Evaluate the limit: $\lim_{x\to 0^+}(1+x)^{1/x}$.

   - Identify the form: as $x \to 0^+$, the base $(1+x) \to 1$ and the exponent $1/x \to \infty$. This is a $1^\infty$ form.
   - Set $y = (1+x)^{1/x}$ and take the natural log: $\ln y = \ln((1+x)^{1/x}) = \frac{1}{x}\ln(1+x)$.
   - Rewrite as a quotient: $\lim_{x\to 0^+} \frac{\ln(1+x)}{x}$. This is a $0/0$ form.
   - Apply L'Hopital's rule: the derivative of $\ln(1+x)$ is $1/(1+x)$ and the derivative of $x$ is 1.
   - Evaluate the limit: $\lim_{x\to 0^+} \frac{1/(1+x)}{1} = 1$.
   - Since $\ln y \to 1$, then $y \to e^1 = e$.

   **Answer:** e

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