L'Hopital's rule I — zero over zero forms

Practice problems

1. Evaluate the limit $\lim_{x\to 0} \frac{x^2}{\sin(x)}$.
2. Evaluate the limit $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$.
3. Evaluate the limit $\lim_{x\to 0} \frac{\tan(x)-x}{x^3}$.

Solutions

1. Evaluate the limit $\lim_{x\to 0} \frac{x^2}{\sin(x)}$.

   - Check the form: as $x \to 0$, $x^2 \to 0$ and $\sin(x) \to 0$. This is a $0/0$ form.
   - Apply L'Hopital's rule: differentiate the numerator to get $2x$ and the denominator to get $\cos(x)$.
   - Evaluate the new limit: $\lim_{x\to 0} \frac{2x}{\cos(x)}$.
   - Substitute $x = 0$: $\frac{2(0)}{\cos(0)} = \frac{0}{1} = 0$.

   Answer: 0

2. Evaluate the limit $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

   - Check the form: $e^0 - 1 - 0 = 0$ and $0^2 = 0$. This is $0/0$.
   - Apply L'Hopital's rule: the derivative of $e^x - 1 - x$ is $e^x - 1$, and the derivative of x^2 is $2x$.
   - Check the new form: $e^0 - 1 = 0$ and $2(0) = 0$. This is still $0/0$.
   - Apply L'Hopital's rule again: the derivative of $e^x - 1$ is e^x, and the derivative of $2x$ is 2.
   - Evaluate the limit: $\lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$.

   Answer: 1/2

3. Evaluate the limit $\lim_{x\to 0} \frac{\tan(x) - x}{x^3}$.

   - Check the form: $\tan(0) - 0 = 0$ and $0^3 = 0$. This is $0/0$.
   - Apply L'Hopital's rule: the derivative of $\tan(x) - x$ is $\sec^2(x) - 1$, and the derivative of x^3 is $3x^2$.
   - Note that $\sec^2(x) - 1 = \tan^2(x)$. The limit is $\lim_{x\to 0} \frac{\tan^2(x)}{3x^2}$.
   - This is still $0/0$. Apply L'Hopital's rule again: the derivative of $\tan^2(x)$ is $2\tan(x)\sec^2(x)$, and the derivative of $3x^2$ is $6x$.
   - The limit is $\lim_{x\to 0} \frac{2\tan(x)\sec^2(x)}{6x} = \lim_{x\to 0} \frac{\tan(x)}{3x} \cdot \sec^2(x)$.
   - Apply L'Hopital's rule one last time to $\frac{\tan(x)}{3x}$: the derivative of $\tan(x)$ is $\sec^2(x)$ and the derivative of $3x$ is 3.
   - The limit becomes $\frac{\sec^2(0)}{3} \cdot \sec^2(0) = \frac{1}{3} \cdot 1 = \frac{1}{3}$.

   Answer: 1/3
