Differentiability vs continuity; corners and cusps

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Practice problems

1. Determine if the function $f(x) = \begin{cases} x^2+1 & x \le 0 \\ x+1 & x > 0 \end{cases}$ is differentiable at $x = 0$.
2. Is the function $f(x) = |2x - 6|$ differentiable at $x = 3$?
3. Explain why $f(x) = x^{1/5}$ is not differentiable at $x = 0$ despite being continuous.

Solutions

1. Determine if the function $f(x) = \begin{cases} x^2+1 & x \le 0 \\ x+1 & x > 0 \end{cases}$ is differentiable at $x = 0$.

   - Check continuity: $f(0) = 0^2 + 1 = 1$. The limit from the right is $0 + 1 = 1$. It is continuous.
   - Find the left-hand derivative: $\lim_{h\to 0^-} \frac{(0+h)^2+1-(0^2+1)}{h} = \lim_{h\to 0^-} \frac{h^2}{h} = 0$.
   - Find the right-hand derivative: $\lim_{h\to 0^+} \frac{(0+h)+1-(0+1)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$.
   - Compare the slopes: $0 \neq 1$.
   - Since the one-sided derivatives are different, the function is not differentiable at $x = 0$.

   Answer: Not differentiable at $x = 0$ (it has a corner).

2. Is the function $f(x) = |2x - 6|$ differentiable at $x = 3$?

   - Check continuity: $f(3) = |2(3) - 6| = 0$. The function is continuous.
   - For $x > 3$, $f(x) = 2x - 6$. The difference quotient is $\frac{2(3+h)-6-0}{h} = \frac{2h}{h} = 2$.
   - For $x < 3$, $f(x) = -(2x-6) = -2x+6$. The difference quotient is $\frac{-2(3+h)+6-0}{h} = \frac{-2h}{h} = -2$.
   - The left-hand slope -2 does not equal the right-hand slope 2.

   Answer: Not differentiable at $x = 3$.

3. Explain why $f(x) = x^{1/5}$ is not differentiable at $x = 0$ despite being continuous.

   - Check continuity: $f(0) = 0^{1/5} = 0$. It is continuous.
   - Set up the difference quotient: $\lim_{h\to 0} \frac{h^{1/5}-0}{h}$.
   - Simplify the expression: $\frac{h^{1/5}}{h} = h^{1/5-1} = h^{-4/5} = \frac{1}{h^{4/5}}$.
   - Evaluate the limit: as $h \to 0$, $1/h^{4/5}$ approaches infinity.
   - Since the slope becomes infinite, there is a vertical tangent.

   Answer: It has a vertical tangent at $x = 0$.
