The Intermediate Value Theorem

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Practice problems

1. Show that the function $f(x) = x^3 + 2x - 5$ has at least one root in the interval $[1, 2]$.
2. Prove that there exists a value c in the interval $[1, 1.5]$ such that $\tan(c) = 2c$. Use the provided values: $\tan(1) \approx 1.557$ and $\tan(1.5) \approx 14.101$.
3. Consider $f(x) = e^x - 3x$. Given that $e \approx 2.718$ and $e^2 \approx 7.389$, show that there are at least two roots in the interval $[0, 2]$.

Solutions

1. Show that the function $f(x) = x^3 + 2x - 5$ has at least one root in the interval $[1, 2]$.

   - Step 1: Note that $f(x)$ is a polynomial, so it is continuous on the closed interval $[1, 2]$.
   - Step 2: Evaluate the function at the left endpoint: $f(1) = 1^3 + 2(1) - 5 = 1 + 2 - 5 = -2$.
   - Step 3: Evaluate the function at the right endpoint: $f(2) = 2^3 + 2(2) - 5 = 8 + 4 - 5 = 7$.
   - Step 4: Since $f(1) = -2$ and $f(2) = 7$, and 0 is between -2 and 7, the Intermediate Value Theorem guarantees there is at least one c in $(1, 2)$ such that $f(c) = 0$.

   Answer: A root exists because $f(1) = -2$ and $f(2) = 7$ and the function is continuous.

2. Prove that there exists a value c in the interval $[1, 1.5]$ such that $\tan(c) = 2c$. Use the provided values: $\tan(1) \approx 1.557$ and $\tan(1.5) \approx 14.101$.

   - Step 1: Define a new function $g(x) = \tan(x) - 2x$. We want to find where $g(c) = 0$.
   - Step 2: Check continuity. $\tan(x)$ is continuous on $[1, 1.5]$ because the first vertical asymptote is at $\pi/2 \approx 1.57$, which is outside this interval.
   - Step 3: Evaluate $g(1) = \tan(1) - 2(1) \approx 1.557 - 2 = -0.443$.
   - Step 4: Evaluate $g(1.5) = \tan(1.5) - 2(1.5) \approx 14.101 - 3 = 11.101$.
   - Step 5: Since $g(1) < 0$ and $g(1.5) > 0$, the IVT guarantees a root in $(1, 1.5)$.

   Answer: A root exists in $(1, 1.5)$ because $g(1) \approx -0.443$ and $g(1.5) \approx 11.101$ and the function is continuous.

3. Consider $f(x) = e^x - 3x$. Given that $e \approx 2.718$ and $e^2 \approx 7.389$, show that there are at least two roots in the interval $[0, 2]$.

   - Step 1: $f(x)$ is continuous on $[0, 2]$ as it is the difference of an exponential and a linear function.
   - Step 2: Evaluate at $x = 0$: $f(0) = e^0 - 3(0) = 1 - 0 = 1$ (positive).
   - Step 3: Evaluate at $x = 1$: $f(1) = e^1 - 3(1) \approx 2.718 - 3 = -0.282$ (negative).
   - Step 4: Evaluate at $x = 2$: $f(2) = e^2 - 3(2) \approx 7.389 - 6 = 1.389$ (positive).
   - Step 5: There is a sign change between $x = 0$ and $x = 1$, so there is a root in $(0, 1)$.
   - Step 6: There is another sign change between $x = 1$ and $x = 2$, so there is a second root in $(1, 2)$.

   Answer: Roots exist in $(0, 1)$ and $(1, 2)$ because the function values change signs twice.
