Infinite limits and vertical asymptotes

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Practice problems

1. Find the vertical asymptote and the one-sided limits as x approaches that asymptote for $f(x) = \frac{4}{x+5}$.
2. Determine the vertical asymptote(s) and the behavior of $f(x) = \frac{1}{(x-2)^2}$ as x approaches the asymptote from both sides.
3. Find all vertical asymptotes for $f(x) = \frac{x-3}{x^2-5x+6}$.

Solutions

1. Find the vertical asymptote and the one-sided limits as x approaches that asymptote for $f(x) = \frac{4}{x+5}$.

   - Set the denominator to zero: $x + 5 = 0$, which gives $x = -5$.
   - Since the numerator is a non-zero constant, $x = -5$ is a vertical asymptote.
   - Test the right-hand limit: for $x = -4.9$, $f(-4.9) = \frac{4}{0.1} = 40$ (positive), so $\lim_{x \to -5^+} f(x) = \infty$.
   - Test the left-hand limit: for $x = -5.1$, $f(-5.1) = \frac{4}{-0.1} = -40$ (negative), so $\lim_{x \to -5^-} f(x) = -\infty$.

   Answer: Vertical asymptote at $x = -5$; $\lim_{x \to -5^+} f(x) = \infty$ and $\lim_{x \to -5^-} f(x) = -\infty$.

2. Determine the vertical asymptote(s) and the behavior of $f(x) = \frac{1}{(x-2)^2}$ as x approaches the asymptote from both sides.

   - The denominator is zero when $(x-2)^2 = 0$, which means $x = 2$.
   - The numerator is 1, so $x = 2$ is a vertical asymptote.
   - Test the right-hand limit: for $x = 2.1$, $(2.1 - 2)^2 = 0.01$, and $1/0.01 = 100$ (positive), so $\lim_{x \to 2^+} f(x) = \infty$.
   - Test the left-hand limit: for $x = 1.9$, $(1.9 - 2)^2 = (-0.1)^2 = 0.01$, and $1/0.01 = 100$ (positive), so $\lim_{x \to 2^-} f(x) = \infty$.

   Answer: Vertical asymptote at $x = 2$; $\lim_{x \to 2^+} f(x) = \infty$ and $\lim_{x \to 2^-} f(x) = \infty$.

3. Find all vertical asymptotes for $f(x) = \frac{x-3}{x^2-5x+6}$.

   - Factor the denominator: $x^2 - 5x + 6 = (x - 2)(x - 3)$.
   - The function is $f(x) = \frac{x-3}{(x-2)(x-3)}$.
   - The value $x = 3$ makes both numerator and denominator zero, so it is a hole.
   - The value $x = 2$ makes only the denominator zero, so it is a vertical asymptote.
   - Simplify the function to $f(x) = \frac{1}{x-2}$ for $x \neq 3$ to confirm the behavior at $x = 2$.

   Answer: The only vertical asymptote is $x = 2$.
