The squeeze theorem (with sin x over x)

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Practice problems

1. Evaluate the limit $\lim_{x\to 0} x^2 \sin\left(\frac{5}{x}\right)$ using the Squeeze Theorem.
2. Given that $2x - 1 \leq f(x) \leq x^2 + 1$ for all x, find $\lim_{x\to 1} f(x)$.
3. Use the result $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ and limit laws to evaluate $\lim_{x\to 0} \frac{\sin(3x)}{x}$.

Solutions

1. Evaluate the limit $\lim_{x \to 0} x^2 \sin\left(\frac{5}{x}\right)$ using the Squeeze Theorem.

   - Start with the known bound for sine: $-1 \leq \sin\left(\frac{5}{x}\right) \leq 1$.
   - Multiply the entire inequality by x^2. Since $x^2 \geq 0$ for all x, the inequality signs remain the same: $-x^2 \leq x^2 \sin\left(\frac{5}{x}\right) \leq x^2$.
   - Evaluate the limit of the lower bound: $\lim_{x \to 0} -x^2 = 0$.
   - Evaluate the limit of the upper bound: $\lim_{x \to 0} x^2 = 0$.
   - Since both bounds approach 0, by the Squeeze Theorem, the middle limit must also be 0.

   Answer: 0

2. Given that $2x - 1 \leq f(x) \leq x^2 + 1$ for all x, find $\lim_{x \to 1} f(x)$.

   - Identify the lower bound function $g(x) = 2x - 1$ and the upper bound function $h(x) = x^2 + 1$.
   - Calculate the limit of the lower bound as x approaches 1: $\lim_{x \to 1}(2x - 1) = 2(1) - 1 = 1$.
   - Calculate the limit of the upper bound as x approaches 1: $\lim_{x \to 1}(x^2 + 1) = 1^2 + 1 = 2$.
   - Observe that the limits are 1 and 2. Since $1 \neq 2$, the Squeeze Theorem cannot be used to find a specific value for the limit.

   Answer: The limit cannot be determined using the Squeeze Theorem because the bounds do not approach the same value.

3. Use the result $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ and limit laws to evaluate $\lim_{x \to 0} \frac{\sin(3x)}{x}$.

   - We want the expression inside the sine to match the denominator. We have $3x$ inside the sine, but only x in the denominator.
   - Multiply the expression by $\frac{3}{3}$ to manipulate the denominator: $\lim_{x \to 0} 3 \cdot \frac{\sin(3x)}{3x}$.
   - Use the constant multiple law to move the 3 outside the limit: $3 \cdot \lim_{x \to 0} \frac{\sin(3x)}{3x}$.
   - As x approaches 0, the term $3x$ also approaches 0. Therefore, $\lim_{x \to 0} \frac{\sin(3x)}{3x} = 1$ based on the fundamental limit proven today.
   - The final result is $3 \cdot 1 = 3$.

   Answer: 3
